Bases for Vector Spaces

Definition

A set of vectors $S = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_n \}$ in a vector space V forms a **basis** for V if

a. S is linearly independent.

b. S spans V.

Examples

$\mathbb{R}^2$ - Any two independent vectors can form a basis in $\mathbb{R}^2$.

Standard Basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$\mathbb{R}^3$ - Any three independent vectors form a basis in $\mathbb{R}^3$.

Standard Basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\vdots$

$\mathbb{R}^n$ - Any n independent vectors form a basis in $\mathbb{R}^n$.

Standard Basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1}, \dots \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1} \right\}$$

Definition

A basis $\{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n \}$ is called **orthogonal** if every pair of vectors is orthogonal (perpendicular). In other words, their dot product must be zero.

$$\vec{v}_i \ \vec{v}_j = 0 \qquad (i \neq j)$$

The following set of vectors are orthogonal.

$$\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 0 \\ 2 \end{pmatrix} \right\}$$

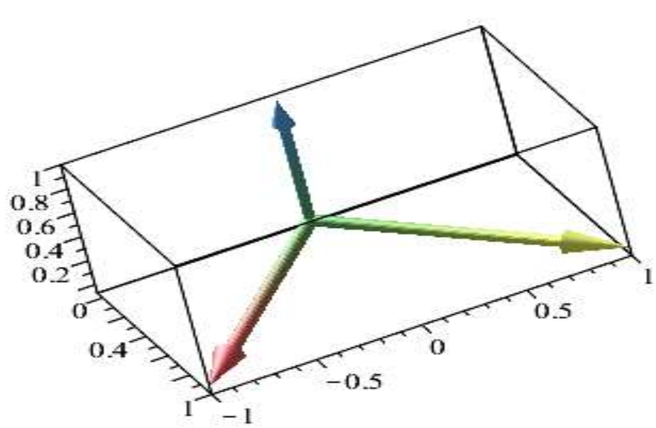
Definition

A basis $\{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n \}$ is called **orthonormal** if the basis is orthogonal and consists of **unit** vectors.

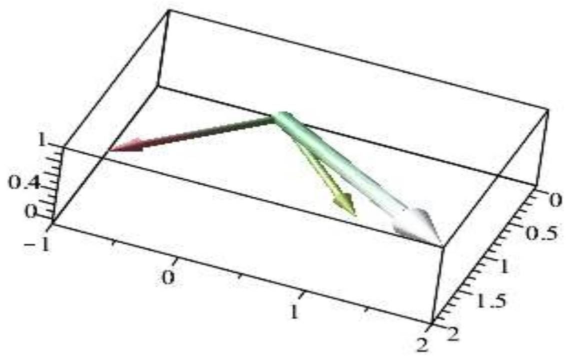
$$\vec{v}_i \cdot \vec{v}_j = 0 \quad (i \neq j) \quad \text{and} \quad \left\| \vec{v}_i \right\|_2 = 1 \quad (i = 1, 2, \ldots, n)$$

Examples

- The vectors $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ form an orthogonal basis (but not orthonormal).



- ♦ The standard basis in $\mathbb{R}^n$ is orthonormal.
- ♦ The vectors $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\}$ form a basis for $\mathbb{R}^3$ which is neither orthogonal nor orthonormal.



Homework

1. Do vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ form a basis for $\mathbb{R}^4$? Explain.
2. Do vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ form a basis for $\mathbb{R}^3$? Explain.
3. Give an example of an orthogonal basis in $\mathbb{R}^2$ other than the standard basis.
4. Give an example of an orthonormal basis in $\mathbb{R}^2$ other than the standard basis.
5. Give an example of an orthogonal basis in $\mathbb{R}^3$ other than the standard basis.
6. Give an example of an orthonormal basis in $\mathbb{R}^3$ other than the standard basis.

Coordinate Systems

Theorem 5

Let the vectors $\{\ \vec{v}_1,\ \vec{v}_2,\ \vec{v}_3,\ \ldots,\ \vec{v}_n\ \}$ be a basis for a vector space V. Then for each vector $\vec{x}$ in V there exists **unique** constants $c_1, c_2, c_3, \ldots, c_n$ such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{x}$$

Proof

Since the vectors form a basis, they span V and there are constants $c_1, c_2, c_3, \ldots, c_n$ such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{x}$. Assume that these constants are not unique, and that there are also constants $b_1, b_2, b_3, \ldots, b_n$ such that $b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n = \vec{x}$. If we subtract one representation from the other, we get

$$(b_1 - c_1)\vec{v}_1 + (b_2 - c_2)\vec{v}_2 + \cdots + (b_n - c_n)\vec{v}_n = \vec{0}$$

Since $\{\ \vec{v}_1,\ \vec{v}_2,\ \vec{v}_3,\ \ldots,\ \vec{v}_n\ \}$ are independent, all the constants $(b_1 - c_1)$, $(b_2 - c_2)\ \ldots$ $(b_n - c_n)$ must be zero, and therefore, $b_1 = c_1,\ b_2 = c_2,\ \ldots,\ b_n = c_n$.

■

Definition

Suppose $\{\ \vec{v}_1,\ \vec{v}_2,\ \vec{v}_3,\ \ldots,\ \vec{v}_n\ \}$ is a basis for a vector space V and $\vec{x}$ is in V. The coordinates of $\vec{x}$ relative to the basis $\{\ \vec{v}_1,\ \vec{v}_2,\ \vec{v}_3,\ \ldots,\ \vec{v}_n\ \}$ are the constants (**weights**) $c_1, c_2, \ldots, c_n$ such that

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

Examples

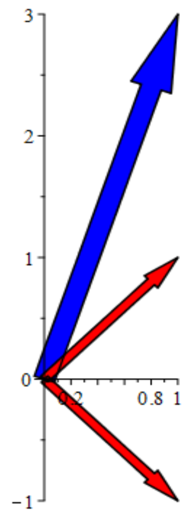
♦ In $\mathbb{R}^n$, when working with the standard basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1},\ \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1},\ \ldots\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}_{n\times 1} \right\}$$

since $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$, then $c_1 = x_1, \ \ldots, \ c_n = x_n$.

However, when working with an arbitrary basis, coordinates change.

♦ Find the coordinates of a the vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ with respect to the basis $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.



We are looking for constants c_1 and c_2 such that

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{(sounds familiar?)}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

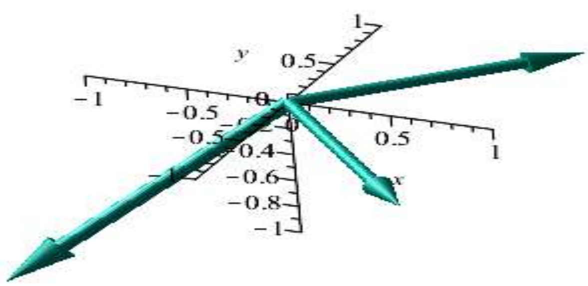
Solution: $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, which are the coordinates of $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ w.r.t. $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

Note: If we want to assign coordinates to all points in $\mathbb{R}^2$ w.r.t. $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$, we must solve the following system.

$$c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$$

Example 2

Find the coordinates of an arbitrary vector $\begin{pmatrix}x\\y\\z\end{pmatrix}$ w.r.t. the basis $\left\{\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\\-1\end{pmatrix}\right\}$.



The coordinates can be obtained by solving the system

$$c_1\begin{pmatrix}1\\-1\\0\end{pmatrix} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix} + c_3\begin{pmatrix}-1\\-1\\-1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}.$$

$$\begin{bmatrix}1 & 1 & -1 & x\\ -1 & 1 & -1 & y\\ 0 & 0 & -1 & z\end{bmatrix} \to \begin{bmatrix}1 & 1 & -1 & x\\ 0 & 2 & -2 & x+y\\ 0 & 0 & -1 & z\end{bmatrix} \to \begin{bmatrix}1 & 1 & -1 & x\\ 0 & 1 & -1 & \dfrac{x+y}{2}\\ 0 & 0 & -1 & z\end{bmatrix} \to$$

$$\begin{bmatrix}1 & 0 & 0 & \dfrac{x-y}{2}\\ 0 & 1 & -1 & \dfrac{x+y}{2}\\ 0 & 0 & 1 & -z\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & \dfrac{x-y}{2}\\ 0 & 1 & 0 & \dfrac{x+y-2z}{2}\\ 0 & 0 & 1 & -z\end{bmatrix}, \quad \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}\dfrac{x-y}{2}\\ \dfrac{x+y-2z}{2}\\ -z\end{pmatrix}$$

Homework

1. Find the coordinates of the vector $\begin{pmatrix}1\\2\\3\end{pmatrix}$ w.r.t. the basis $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}2\\2\\7\end{pmatrix}\right\}$.

2. Find the coordinates of the vector $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ w.r.t. the basis $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\2\\2\end{pmatrix}\right\}$.